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Moment Problem and Spectral Theorem

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Hausdorff momentum problem and its relations to spectral theorem for bounded Hilbert space operators are treated. A generalization for some ordered algebras is shown, where projections are replaced by idempotents.

KEY WORDS: Hausdorff momentum problem; spectral theorem; ordered algebra; completely monotone sequence.

1. INTRODUCTION

Recall that the classical Hausdorff problem is the following: given a set of real numbers $\{v_n\}_{n=0}^{\infty}$, find a probability measure μ on the Borel subsets of the unit interval [0, 1] such that

$$v_n = \int_0^1 t^n \, d\mu(t), \quad n \geq 0.$$

Hausdorff has shown that the answer is positive iff the sequence is completely monotone (see Definition 1 below) (Hausdorff, 1921a,b, 1923). For a review of the momentum problem see Shohat and Tamarkin (1943) and Widder (1946), for results on the Hilbert space operators see Riesz and Sz.-Nagy (1955). The Hausdorff momentum problem in the context of effect algebras has been treated in Duchoň *et al.* (1997).

In the present paper, we relate the momentum problem to the spectral theorem. In the first part, we give a detailed solution of the momentum problem for bounded self-adjoint (s.a.) operators on a Hilbert space. In comparison with Duchoň *et al.* (1997), more direct methods are applied. Then we show that the solution of the momentum problem yields an alternative proof of the spectral theorem. Finally, we extend the results to some ordered algebras.

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2. HAUSDORFF MOMENTUM PROBLEM FOR HILBERT SPACE OPERATORS AND SPECTRAL THEOREM

Let H be a Hilbert space, $L(H)$ the lattice of all closed subspaces of H, $\mathcal{S}(H)$ the set of all (bounded) s.a. operators and $\mathcal{E}(H)$ the effect algebra of all s.a. operators A with $0 \leq A \leq I$. By an observable we mean a POV measure on Borel subsets of the interval $[a, b]$ of real line, a state is a probability measure on $L(H)$. By Gleason's theorem, states correspond to positive operators D with trace 1 such that

$$m(P) = \mathrm{tr} DP, \quad P \in L(H).$$

Hence, states correspond to the positive linear functionals on the algebra $\mathcal{B}(H)$ of all bounded operators on H of the form $B \mapsto \mathrm{tr} DB$. The restriction of a state to $\mathcal{E}(H)$ is a state on $\mathcal{E}(H)$. Observe that if s is a state and y is an observable, then $E \rightarrow s(y(E))$ is a usual probability measure on $B([a, b])$.

We shall say that a sequence $(a_n)_0^\infty$ of elements of $\mathcal{S}(H)$ is a *solution of the observable momentum problem* if there is an observable (POV measure) $y : B([a, b]) \rightarrow \mathcal{S}(H)$ such that

$$s(a_n) = \int_a^b t^n s(y(dt))$$

for every state s.

First we take for $[a, b]$ the unit interval $[0, 1]$.

Definition 1. Let $(v_n)_0^\infty$ be a sequence of numbers. Define, for $k = 0, 1, 2, \ldots,$ the operator Δ^k by

$$\Delta^0 v_n = v_n, \qquad \Delta^1 v_n = v_n - v_{n+1}$$
$$\Delta^k v_n = v_n - \binom{k}{1} v_{n+1} + \binom{k}{2} v_{n+2} + \cdots + (-1)^k v_{n+k}, \quad n = 0, 1, \ldots.$$

We say that the sequence $(v_n)_0^\infty$ is *completely monotone* if $\Delta^k v_n \geq 0$, where $n, k = 0, 1, \ldots.$

Now Hausdorff momentum theorem in the classical probability theory says that for a sequence $(v_n)_0^\infty$ to be the moment sequence of some unique positive measure μ on $[0, 1]$ it is necessary and sufficient that $(v_n)_0^\infty$ be completely monotone. In the following theorem we extend this result to the POV-observable momentum problem. First we need the following definition.

Definition 2. Let $(a_n)_0^\infty$ be a sequence of effects. We shall say that this sequence is *completely monotone* if for every vector state s_ψ, $(\psi \in H, \|\psi\| = 1)$ the sequence $(s_\psi(a_n))_0^\infty$ is completely monotone.

Since every state can be expressed as a convex combination of vector states, a sequence $(a_n)_0^\infty$ of effects is completely monotone if and only if for every state s the sequence $(s(a_n))_0^\infty$ is completely monotone.

Theorem 1. *A sequence of effects $(a_n)_0^\infty$ is a solution of the POV-observable momentum problem, i.e., there is a POV measure $y : B([0, 1]) \to \mathcal{E}(H)$ such that*

$$a_n = \int_0^1 t^n y(dt), \quad n = 0, 1, \dots.$$

if and only if $(a_n)_0^\infty$ is completely monotone.

Proof: Let there exist an observable y such that

$$a_n = \int_0^1 t^n y(dt), \quad n = 0, 1, \dots,$$

that is,

$$\langle a_n\psi, \psi\rangle = \int_0^1 t^n \langle y(dt)\psi, \psi\rangle$$

for every unit vector $\psi \in H$. Then as $E \mapsto \langle y(E)\psi, \psi\rangle$ is a probability measure on $B([0, 1])$, $(s_\psi(a_n))_0^\infty$ is a solution of the classical momentum problem for this probability measure, and hence this sequence is completely monotone.

Conversely, assume that $(a_n)_0^\infty$ is completely monotone. Then for every unit vector $\psi \in H$, the sequence $(\langle a_n\psi, \psi\rangle)_0^\infty$ is completely monotone. Therefore, by the classical result, there is a measure μ_ψ on $B([0, 1])$ such that

$$\langle a_n\psi, \psi\rangle = \int_0^1 t^n \mu_\psi(dt), \quad k = 0, 1, \dots.$$

Since the effects a_n are nonnegative operators, the mapping $\psi \mapsto \langle a_n\psi, \psi\rangle$ is a nonnegative quadratic form on H for every n.

For any $\phi, \psi \in H$ we then have, for every $n = 0, 1, \dots,$

$$\int_0^1 t^n \mu_{\psi+\phi}(dt) + \int_0^1 t^n \mu_{\psi-\phi}(dt) = 2\left(\int_0^1 t^n \mu_\psi(dt) + \int_0^1 t^n \mu_\phi(dt)\right).$$

Using Weierstrass theorem, t^n can be replaced by f for any continuous function f defined on $[0, 1]$. This implies that

$$\mu_{\psi+\phi}(E) + \mu_{\psi-\phi}(E) = 2(\mu_\psi(E) + \mu_\phi(E))$$

for every $E \in B([0, 1])$.

Similarly we prove that

$$\sqrt{\mu_{\psi+\phi}}(E) \leq \sqrt{\mu_\psi}(E) + \sqrt{\mu_\phi}(E).$$

This shows that $\psi \mapsto \sqrt{\mu_\psi(E)}$ is a seminorm satisfying the parallelogram law. Therefore, there is a symmetric sesquilinear functional $(\psi, \phi) \mapsto \nu_E(\psi, \phi)$ such that $\nu_E(\psi, \psi) = \mu_\psi(E)$. It follows by Riesz theorem that there is an s.a. operator $y(E)$ such that $\nu_E(\psi, \phi) = \langle y(E)\psi, \phi\rangle$. Since μ_ψ is a probability measure, it is clear that

$$0 \leq \mu_\psi(E) = \langle y(E)\psi, \psi\rangle \leq 1$$

for every ψ, hence $y(E)$ is an effect and the mapping $E \mapsto y(E)$ is a POV measure.

We obtain that, for every $\psi \in H$,

$$s_\psi(a_n) = \int_0^1 t^n \langle y(dt)\psi, \psi\rangle = \int_0^1 t^n s_\psi(y(dt)),$$

which can be written as

$$a_n = \int_0^1 t^n y(dt). \quad \square$$

Now we will show that the solution of Hausdorff momentum problem yields an alternative proof of spectral theorem for Hilbert space effects.

Let $A \in \mathcal{E}(H)$, i.e. A is an s.a. operator with $0 \leq A \leq I$.

We have

$$\Delta^0 A^n = A^n, \qquad \Delta^1 A^n = A^n - A^{n+1} = (I - A)A^n$$

$$\Delta^k A^n = A^n - \binom{k}{1} A^{n+1} + \binom{k}{2} A^{n+2} + \cdots + (-1)^k A^{n+k}$$

$$= (I - A)^k A^n, \quad n = 0, 1, \ldots.$$

It is known that the product of two commuting positive operators is a positive operator. From the assumption $0 \leq A \leq I$ it follows that $\{A^n\}$ is completely monotone. Hence there exists a unique *POV* measure $y : B([0, 1]) \to \mathcal{E}(H)$ such that

$$A^n = \int_0^1 t^n y(dt), \quad n = 0, 1, \ldots. \tag{1}$$

Now we would like to show that the set of values $y(E)$, $E \in B([0, 1])$, is a system of idempotents on H. For any two polynomials p and q we have, using elementary "functional calculus" for A (Berberian, 1966)

$$p(A)q(A) = \int_0^1 p(t)q(t)y(dt)$$

Let f be any real-valued continuous function on $[0, 1]$. By Weierstrass theorem, there is a sequence of polynomials $\{p_n\}$ uniformly converging to f. Put

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}.$$

It follows

$$\begin{aligned}\|p(A)\| &= \sup\{|\langle p(A)\psi, \psi\rangle| : \psi \in H, \|\psi\| = 1\}\\ &= \sup_{\|\psi\|=1} \int_0^1 |p(t)|\langle y(dt)\psi, \psi\rangle\\ &\leq \|p\|_\infty \int_0^1 \langle y(dt)\psi, \psi\rangle \leq \|p\|_\infty.\end{aligned}$$

Therefore

$$\|p_n(A) - p_m(A)\| \leq \|p_n - p_m\|_\infty \to 0,$$

and by the norm-completeness of $\mathcal{B}(H)$, there is an s.a. operator $f(A)$ such that $\|p_n(A) - f(A)\| \to 0$. It can be easily checked that $f(A)$ does not depend on the choice of the sequence $\{p_n\}$. Let f, g be any continuous functions on [0, 1] and $\{p_n\}$ and $\{q_n\}$ sequences of polynomials converging uniformly to f and g respectively, then

$$\begin{aligned}&\|p_n(A)q_n(A) - f(A)g(A)\|\\ &\quad\leq \|p_n(A)q_n(A) - f(A)q_n(A)\| + \|f(A)q_n(A) - f(A)g(A)\|\\ &\quad\leq \|q_n(A)\| \cdot \|p_n(A) - f(A)\| + \|f(A)\| \cdot \|q_n(A) - g(A)\|\\ &\quad\leq K\|p_n(A) - f(A)\| + \|f(A)\| \cdot \|q_n(A) - g(A)\| \to 0, \quad n \to \infty,\end{aligned}$$

where K is a bound of the sequence $\{q_n\}$.

Let C be a closed subset of [0, 1], χ_C its characteristic function. By Halmos (1950), there is a monotone sequence $\{f_n\}$ of continuous functions decreasing to χ_C. Defining $g_n = f_n^2$, we have also $g_n \downarrow \chi_C$. By monotone convergence theorem, $\int_0^1 f_n\langle dy\,\psi, \psi\rangle \to \int_0^1 \chi_C\langle dy\,\psi, \psi\rangle$ for every $\psi \in H$. That is, $\int_0^1 f_n\,dy \to \int_0^1 \chi_C\,dy = y(C)$ strongly, $\int_0^1 g_n\,dy \to \int_0^1 \chi_C\,dy$ strongly. That is $f_n(A) \to y(C)$ strongly, $g_n(A) \to y(C)$ strongly. Since $g_n(A) = f_n(A)^2$ by the previous paragraph, we have for any vector $\psi \in H$,

$$\begin{aligned}\langle y(C)\psi, \psi\rangle &= \lim_{n\to\infty}\langle g_n(A)\psi, \psi\rangle\\ &= \lim_{n\to\infty}\langle f_n(A)\psi, f_n(A)\psi\rangle = \langle y(C)\psi, y(C)\psi\rangle\end{aligned}$$

by the continuity of inner product. Thus $y(C) = y(C)^2$ for any closed subset C of [0, 1]. By regularity of POV measures (Berberian, 1966), $y(E)$ is an idempotent for every Borel subset of [0, 1].

More generally, if A is an operator in $\mathcal{S}(H)$ such that $aI \leq A \leq bI$, $a < b$, we put

$$B = \frac{A - aI}{b - a}.$$

Then B is an s.a. operator with $0 \leq B \leq I$. If we use the substitution in (1) for B we obtain a representation for the successive powers of A:

$$A^n = \int_a^b t^n z(dt), \quad n = 0, 1, \ldots$$

where $z(\ )$ is an observable with the same properties as $y(\ )$, but corresponding to the interval $[a, b]$.

We have thus proved the following.

Theorem 2. *If A is an s.a. operator in $\mathcal{S}(H)$ such that $aI \leq A \leq bI$, then there exists a unique observable $y : B(\mathbb{R}) \to \mathcal{S}(H)$ taking values in a commutative family of idempotents such that*

$$A = \int_a^b t y(dt),$$

with y satisfying $y(\emptyset) = 0$, $y([a, b]) = I$.

3. A GENERALIZATION TO SOME ORDERED ALGEBRAS

Let $\mathcal{R}$ be an algebra with unit 1 such that $(\mathcal{R}, +)$ forms a partially ordered abelian group with a positive cone $\mathcal{R}^+$, and the following conditions are satisfied for every $a, b \in \mathcal{R}^+$:

(R1) If $ab = ba$ then $ab \in \mathcal{R}^+$.
(R2) $a(ba) = (ab)a = aba$ and $aba \in \mathcal{R}^+$.
(R3) $aba = 0$ implies $ab = ba = 0$.
(R4) $(a - b)^2 \in \mathcal{R}^+$.
(R5) $1 \in \mathcal{R}^+$.

That is, $\mathcal{R}$ is an "effect ring" in the terminology of Foulis (2000).

Observe that for any $a, b, c \in \mathcal{R}^+$, c commuting with both a and b, we get by (R1)

$$a \leq b \Rightarrow ac \leq bc. \tag{2}$$

For every $a \in \mathcal{R}$ we may define powers by induction, i.e. we put

$$a^0 = 1, a^1 = a, a^n = a^{n-1} \cdot a, \quad n > 1.$$

An element $p \in \mathcal{R}^+$ is an idempotent (or a projection) if $p^2 = p$. We have by (R4) that $1 - p \in \mathcal{R}^+$, so that $1 - p$ is also an idempotent. It has been proved in Greechie *et al.* (1995), that p is an idempotent if and only if $p \wedge (1 - p) = 0$, and that the set $P(\mathcal{R})$ of all idempotents forms an orthomodular poset.

We will say that an ordered algebra $\mathcal{R}$ is *Dedekind σ-complete* if for every monotone increasing sequence of its elements that is bounded from above the supremum exists.

Property (R2) implies that for any $a \in \mathcal{R}^+$ and any $j, k \in \mathbb{N}$, $a^j a^k = a^k a^j = a^{j+k}$.

For a sequence $\{a_n\}$ of elements of an ordered algebra $\mathcal{R}$ we define

$$\Delta^0 a_n = a_n,$$
$$\Delta^k a_n = a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} + \cdots + (-1)^k a_{n+k}, \quad n = 0, 1, \ldots$$

Definition 3. We will say that a sequence $\{a_n\}$ of elements of an ordered algebra $\mathcal{R}$ is *completely monotone* if $\Delta^k a_n \geq 0$ for every $n, k = 0, 1, \ldots$

Proposition. *Let $\mathcal{R}$ be an ordered algebra. Then for every element $a \in \mathcal{R}$, $0 \leq a \leq 1$, the sequence $\{a^n\}$ is completely monotone.*

Proof: Owing to (R1) and (R2),

$$\sum_{j=1}^{k} (-1)^j \binom{n}{k} a^{n+j} = a^n (1 - a)^k \geq 0. \quad \square$$

A *state* on $\mathcal{R}$ is a bounded positive real linear functional that evaluates the unit $1 \in \mathcal{R}$ by 1. Recall that a state m on $\mathcal{R}$ is *normal* if $a_n \uparrow a$ implies $m(a_n) \to m(a)$.

A set M of states on $\mathcal{R}$ is *ordering* if

$$\mathcal{R}^+ = \{a \in \mathcal{R} : m(a) \geq 0 \; \forall m \in M\}.$$

If M is ordering it is *separating,* i.e. if $m(a) = m(b)$ for all $m \in M$ then $a = b$. Indeed, we have $m(b - a) = m(a - b) = 0$ for all $m \in M$, hence $b - a \in \mathcal{R}^+$ and $a - b \in \mathcal{R}^+$, hence $a = b$. If M is ordering, the convex envelope of M is ordering as well, so we may assume that M is a convex set.

An observable on $\mathcal{R}$ is a mapping $x : \mathcal{B}(\mathbb{R}) \to \mathcal{R}^+$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the real line, such that

(O1) $x(\mathbb{R}) = 1$,
(O2) if $E \cap F = \emptyset$ then $x(E \cup F) = x(E) + x(F)$,
(O3) if $E_i \uparrow E$ then $x(E) = \bigvee_{i=1}^{\infty} x(E_i)$

An observable x is monotone, that is, $E \subseteq F$ implies $x(E) \leq x(F)$. Owing to $x(\mathbb{R}) = 1$, the range of x is contained in the set $\{a \in \mathcal{R} : 0 \leq a \leq 1\}$, that is, in the effect algebra corresponding to $\mathcal{R}$. We will say that an observable x on $\mathcal{R}$ is *projection valued* (PV) if its range is contained in $P(\mathcal{R})$.

If x is an observable and m is a state on $\mathcal{R}$, then the composite mapping $m \circ x : \mathcal{B}(\mathbb{R}) \to [0, 1]$ is a probability measure. If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, then $x \circ f^{-1}$ is also an observable. We can define the expectation of the observable $x \circ f^{-1}$ in a state m in the usual way, and by the integral transformation theorem we get

$$m(x \circ f^{-1}) = \int_{\mathbb{R}} f(t)\, m \circ x(dt) = \int_{\mathbb{R}} t m \circ f^{-1}(dt).$$

Definition 4. Let M be an ordering set of states on an ordered algebra $\mathcal{R}$. We will say that an element $a \in \mathcal{R}$ has a *spectral decomposition* if there is a PV observable y_a such that for every $m \in M$,

$$m(a) = \int_{\mathbb{R}} t m \circ y_a(dt).$$

The following so-called existence property has been introduced in Duchoň *et al*. (1997).

Definition 5. We say that $\mathcal{R}$ has the *existence property* if for every convexity preserving mapping $\nu : M \to M(\mathcal{B}([0, 1]))$ there is an observable y such that for every $m \in M$, $E \in \mathcal{B}([0, 1])$,

$$m(y(E)) = \nu(m)(E),$$

where $M(\mathcal{B}([0, 1])$ is the set of all probability measures on Borel subsets of $[0, 1] \subset \mathbb{R}$.

For example, every von Neumann algebra with no Type I_2 direct summand has the existence property.

Our main result in this section is the following theorem.

Theorem 5. *Let $\mathcal{R}$ be a Dedekind monotone σ-complete ordered algebra with an ordering set of normal states M, having existence property with respect to M. Then every $a \in \mathcal{R}$, $0 \leq a \leq 1$, has a spectral decomposition.*

Proof: Let $a \in \mathbb{R}$, $0 \leq a \leq 1$. By Proposition, the sequence $\{a_n\}$ is monotone. For every positive linear functional m,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} m(a^{k+j}) = m\left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} a^{k+j}\right) = m(a^j(1-a)^k) \geq 0,$$

hence $\{m(a^n)\}$ is a completely monotone sequence. By the solution of classical momentum problem, there is a measure μ_m on $\mathcal{B}([0, 1])$ such that for every $n \in \mathbb{N}$ and every $m \in M$,

$$m(a^n) = \int_0^1 t^n \mu_m(dt).$$

The map $m \mapsto \mu_m$ is convexity preserving; therefore, by the existence property, there is an observable $y : \mathcal{B}([0, 1]) \to \mathcal{R}$ such that

$$\mu_m(E) = m(y(E)), \quad E \in \mathcal{B}([0, 1]).$$

If $p(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_1 t + \alpha_0,$ define

$$p(a) = \alpha_n a^n + \alpha_{n-1} a^{n-1} + \cdots + \alpha_1 a + \alpha_{0.1},$$

If p and q are polynomials, then $(p \cdot q)(a) = p(a)q(a)$, by the commutativity of a^n, a^m.

Let f be a continuous function on $[0, 1]$ and p_n a sequence of polynomials converging uniformly to f. Without loss of generality we may assume that $\|p_n - p_{n-1}\|_\infty \leq 2^{-n}$. Define polynomials

$$q_n = p_n - 2^{-n}.$$

We have

$$q_n - q_{n-1} = p_n - p_{n-1} + 2^{-n} \geq -\|p_n - p_{n-1}\|_\infty + 2^{-n} \geq 0.$$

Moreover,

$$\|f - q_n\|_\infty = \|f - p_n + 2^{-n}\|_\infty \leq \|f - p_n\|_\infty + 2^{-n} \to 0.$$

It follows that $\{q_n\}$ is an increasing sequence uniformly converging to f.

For every $m \in M$ we have

$$\int q_n(t) d\mu_m(t) = \int q_n(t) m \circ y(dt) \to \int f(t)\, d\mu_m(t)$$

for every $m \in M$. Therefore $m(q_n(a)) \to \int f(t)\, d\mu_m(dt)$ for every $m \in M$.

Moreover,

$$m(q_n(a)) - m(q_{n-1}(a)) = \int_0^1 (q_n(t) - q_{n-1}(t)) m \circ y(dt) \geq 0.$$

It follows that the sequence $\{q_n(a)\}$ is monotone increasing, and is bounded from above by $K.1$, where $K \geq 0$ is the bound of f. By Dedekind σ-completeness, there is an element u in $\mathcal{R}$ equal to the supremum of the sequence. Put $u =: f(a)$. By normality of the states $m \in M$, $m(f(a)) = \lim_{n\to\infty} m(q_n(a))$. Moreover, for every $m \in M$,

$$\left| \int q_n(t)\, d\mu_m(t) - \int f(t)\, d\mu_m(t) \right| \leq \int |q_n(t) - f(t)|\, d\mu_m(t) \to 0.$$

It follows that

$$m(f(a)) = \int f(t)\, d\mu_m(t) = \int tm(y \circ f^{-1}(dt))$$

for every $m \in M$.

Let $f_n \downarrow f, g_n \downarrow g, 0 \leq f_n, g_n \leq 1$ being sequences of continuous functions, f, g are bounded measurable functions on [0, 1]. Again by Dedekind monotone σ-completeness we find elements $f(a)$, $g(a)$ as limits of monotone decreasing sequences of elements $f_n(a), g_n(a)$, where for every n, $0 \leq f_n(a) \leq 1$, $0 \leq g_n(a) \leq 1$. We have, by (2) and monotonicity of states,

$$\begin{aligned} &|m(f_n(a)g_n(a) - f(a)g(a))| \\ &\quad \leq |m(f_n(a)g_n(a)) - m(f(a)g_n(a))| + |m(f(a)g_n(a)) - m(f(a)g(a))| \\ &\quad \leq |m(f_n(a)) - m(f(a))| + |m(g_n(a)) - m(g(a))| \to 0. \end{aligned}$$

Now let C be a closed subset of [0, 1], χ_C its characteristic function. There is a sequence of continuous functions f_n between 0 and 1 decreasing to χ_C. Then for every $m \in M$,

$$m(f_n(a)) = \int f_n(t)\, d\mu_m(t) \to \chi_C(t)\, d\mu_m(t).$$

If $g_n = f_n^2$, then $g_n(t) \downarrow \chi_C(t)$.

So $f_n^2(a) \to (\chi_C(a))^2 = \chi_C(a)$. Hence $y(C) = \chi_C(a)$ is a projection. □

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